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LETTER TO THE EDITOR

On the zero-curvature representation of a new integrable system in three dimensions

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Abstract. We have obtained a new integrable equation in three dimensions, by starting from a zero-curvature equation written in terms of gauge fields. It is also demonstrated that a recent attempt at obtaining the zero-curvature representation of the KP equation is erroneous.

The existence of the zero-curvature equation for integrable systems in two dimensions is an established fact [1]. On the other hand a similar result is not known in the case of three-dimensional systems. Here in this letter we show that if we start from a zero-curvature representation in terms of gauge fields, then by a proper choice of the components of the fields we can arrive at a non-trivial nonlinear system in three dimensions.

Let us start from a gauge field $A_\mu^\alpha(x, y, t)$, where the index α is an SU(2) label and μ stands for the spacetime component. The zero-curvature equation for A_μ^α is [2]

$$\partial_\nu A_\mu^\alpha - \partial_\mu A_\nu^\alpha = \varepsilon_{abc}[A_\mu^b, A_\nu^c] \tag{1}$$

which, written in full, reads

$$\begin{aligned} &\partial_t A_x^0 - \partial_x A_t^0 = A_t^- A_x^+ - A_x^- A_t^+ \\ &\partial_t A_x^- - \partial_x A_t^- = 2(A_t^0 A_x^- - A_x^0 A_t^-) \\ &\partial_t A_x^+ - \partial_x A_t^+ = 2(A_x^0 A_t^+ - A_t^0 A_x^+) \end{aligned} \tag{2}$$

along with two other similar sets of equations obtained by cyclic change of (x, t) to (y, x) and (t, y). So in total we have nine equations. Let us now assume that the coordinates (x, y, t) have scale dimension $(-a, -b, -c)$ respectively. So comparing the scale dimensions of left- and right-hand sides of equation (2) we arrive at the conclusion that the following are possible assigned values of the scale dimensions for the gauge fields:

$$\begin{array}{lll} A_x^- \to [0] & A_y^- \to [b-a] & A_t^- \to [c-a] \\ A_x^0 \to [a] & A_y^0 \to [b] & A_t^0 \to [c] \\ A_x^+ \to [2a] & A_y^+ \to [b+a] & A_t^+ \to [c+a] \end{array}$$

of which dimension of at least one is by choice (in this case A_x^-). If we now set

$$A_x^- = 1/2\xi \qquad \text{where } \xi \text{ is a constant} \tag{3}$$

then from equation (2) we at once deduce

$$A_t^0=\xi(-\partial_x A_t^- + 2A_x^0 A_t^-) \qquad A_y^0=\xi(-\partial_x A_y^- + 2A_x^0 A_y^-)$$
$$A_t^+=2\xi(-\xi\partial_x^2 A_t^- + 2\xi(\partial_x A_x^0)A_t^- + 2\xi A_x^0\partial_x A_t^- - \partial_t A_x^0 + A_t^- A_x^+) \tag{4}$$
$$A_y^+=2\xi(-\xi\partial_x^2 A_y^- + 2\xi A_y^-\partial_x A_x^0 + 2\xi A_t^0\partial_x A_y^- - \partial_y A_x^0 + A_y^- A_x^+).$$

So we have four independent gauge field components, which we choose as follows:

$$A_x^+ = A_t^- = u \qquad \text{with scaling dimension } [2a]$$
$$A_x^0 = A_y^- = v \qquad \text{with scaling dimension } [a].$$

From equations (2) and (4) it then follows that

$$u_t + 2\xi^2 u_{xxx} - 4\xi^2 u v_{xx} + 2\xi v_{xt} - 8\xi^2 u_x v_x - 2\xi(2\xi+1)uu_x$$
$$- 8\xi^2 v^2 u_x - 8\xi^2 uvu_x + 4\xi v v_t = 0 \tag{5}$$
$$u_y + 2\xi^2 v_{xxx} - 4\xi^2 v v_{xx} - 2\xi v_{xy} - 2\xi^2 v u_x - 2\xi(\xi+1)uv_x$$
$$+ 4\xi(1-\xi)uv^2 - 8\xi v_x^2 + 4\xi v v_y - 16\xi^2 v^2 v_x = 0 \tag{6}$$
$$u_y - v_t + 2\xi(uv_x - u_x v) = 0. \tag{7}$$

Other equations arising out of the set (2) are seen to be equivalent to these. Equations (5), (6), (7) are the new coupled three-dimensional nonlinear equations for u and v. Now we substitute

$$u = f(\eta) \qquad v = g(\eta) \qquad \eta = x + \lambda y + \mu t$$

whence we get

$$(\lambda - 2\xi g)f' = (\mu - 2\xi f)g'$$

which upon integration and the choice $\nu = \mu/\lambda$ leads to $f = \nu g$. So finally equations (5), (6), (7) reduce to an ordinary nonlinear equation that can be written as

$$2\mu\xi g'' + \nu^2\xi^2 g g' = 2\xi(1-\xi)g^3. \tag{8}$$

It is interesting to observe that equation (8) belongs to the class of equations originally studied by Painlevé and falls in the category subcase i(b) discussed in Ince [3].

Lastly it may be mentioned that any other choice of the gauge fields makes the equations trivial and the whole problem reduces to the old KdV case in two dimensions. For example, in a recent preprint [4] Das *et al* tried to deduce the zero-curvature representation starting from (1). But in addition to the choice (3) they also imposed the condition that $v = \alpha_1$, where α_1 is a constant. Now from equation (7) it is obvious that in that case one obtains

$$u_y - 2\xi\alpha_1 u_x = 0 \tag{9}$$

which immediately implies that $u = f(2\xi\alpha_1 y - x)$, where f is an arbitrary function; as such an equation in (x, y, t) for the function u is actually a two-dimensional one. As such the KP equation in [4] is actually a KdV in a new space variable, because an equation of the type (9) has been used to deduce the equation. But in our case of the coupled (u, v) system there is no such restriction either on u or v.

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